Proofs in Analysis

no step left behind

Diego Cortez
diego@mathisart.org
mathisart.org

(Chapter) Preface

(subsection) What is mathematics?

Mathematics is the **exercise** of **reason**. (And the study of reason itself.)

To do mathematics is **to exercise** our reason.
To exercise our reason is **to do** mathematics.

Mathematics explores the **math realm**, and discovers **proofs**.

(subsection) The cornerstone of mathematics

Truth is the cornerstone of mathematics. Without truth, there is no mathematics.
Truth is the cornerstone of the Heavens and the Earth. The **Lord God Jesus Christ of Nazareth** *is* The Truth.

I like to think the "goal" of mathematics is **to find logical truths**. (There are higher truths that can't be accessed with rational thought.)
How do we go from one logical truth to the next? Via **proof**.

Proof is the lifeblood of mathematics, connecting truth to truth.

Come to think of it, maybe the "goal" of math is *not* to find logical truths, but **to find proofs**... since logical **truth** is often inaccessible to math (in part due to incompleteness, nonconstructibility, uncertainty, undecidability, incomputability, ...).
Not all that is **true** *can be* **proven**, (**incompleteness**)
not all that **exists** *can be* **shown**. (**nonconstructibility**)

(subsection) Two kinds of proof

There are two kinds of proofs: **formal proofs** and **"social" proofs**.

A **formal proof** is a mechanical *tree* of (logical) **sentences**.
The *nodes* of the tree (ie. the sentences) are connected by **deduction**.
The *root* of the tree is the sentence that we're proving.
Formal proofs are **rigorous**.

A **"social" proof** is a flabby argument for why a (logical) sentence *may* be **true**.
"Social" proofs give us a **rough idea** of why a sentence *may* be **true**.
"Social" proofs *rarely* give us a **good idea** in practice, since most of them **skip lots of steps** (or worse: they leave them as "*exercise*").
"Social" proofs are what we find in most textbooks (like this one).
"Social" proofs are **not rigorous**, by their vagueness and incompleteness.

Formal proofs are the **machine code** of mathematics.
"Social" proofs are the **natural language** of mathematics.

(subsection) Skipping steps is bad mathematics

There's *exactly one* trivial thing in math: **skipping steps**.

It's easy to "prove" something when we **skip steps**. For example,

THEOREM. The Riemann hypothesis.
PROOF. Exercise.

A proof that skips steps is **no proof at all**. Just as the mathematics community shouldn't accept proofs with holes, a math student should *never* accept a proof with holes.

Yet, my experience is that proofs in textbooks are often full of holes:
stuff that is assumed,
stuff that is ambiguous,
stuff that is unclear,
stuff that is left to the reader,
stuff that is left as "exercise",
stuff that is left to context,
stuff that depends on stuff that hasn't been proved,
stuff that depends on itself (circularity),
stuff that is simply ignored.
All this makes for bad explanations. Good mathematics is pristine, precise. Bad explanations are bad mathematics.

It takes intelligence to communicate clearly. It's trivial to speak gobbledygook that others don't understand.

The **burden of explanation** is on the teacher/writer, *not* on the student/reader!
A good doctor doesn't tell patients to "treat themselves". A student's job is *not* to "convince himself".
It's the *responsibility* of the teacher/writer to make himself understood. If he's not understood, then **he has failed**. Badly.

(subsection) Proofs: the good, the bad, and the awesome

A **good proof** is a proof where **every step is "easy" to follow**, *and* **no step is skipped**.

A **bad proof** is a proof where **some steps are hard to follow**, *or* **some steps are skipped**.

The *hallmark* of a **good proof** is that **the reader doesn't need to do any work** to follow the proof.
In particular, the **reader doesn't need to stop and think** about some step, *and* he **doesn't need pen and paper to follow the proof** (the **writer** has supplied all steps/calculations).

The *hallmark* of a **bad proof** is that **the reader needs to do some work** to follow the proof.
In particular, the **reader needs to stop and think** about some step, *or* he **needs pen and paper to follow the proof** (the **writer** has skipped some steps/calculations).

An **awesome proof** is a *good proof* that's also **at the right level of abstraction**.
If the proof is too low-level, it'll be hard to *aggregate the details* into the high-level ideas of the proof.
If the proof is too abstract, it'll be hard to *specialize the generalities* into the details of the proof.

Reading and understanding **awesome proofs** is hard.
Reading and understanding **good proofs** is very hard.
Reading and understanding **bad proofs**... is near-impossible.

(subsection) The proofs in this book

The proofs in this book are *not* **good**, let alone **awesome**. But I'm not actively trying to make them **bad**.

(Chapter) Classical logic

Mathematics is the discovery of proofs.
Mathematics is the exercise of reason.
Reason is the exercise of logic (maybe?).

AXIOM. Creating new propositions from old propositions via logical connectives.
0) For all propositions $P, Q \langle$ NOT P is a proposition$\rangle$.
1) For all propositions $P, Q \langle P$ AND Q is a proposition$\rangle$.
2) For all propositions $P, Q \langle P$ OR Q is a proposition$\rangle$.
3) For all propositions $P, Q \langle P$ THEN Q is a proposition$\rangle$.
4) For all propositions $P, Q \langle P$ IFF Q is a proposition$\rangle$.
AXIOM. The law of the **excluded middle**. For every proposition $P \langle$the proposition $P \vee \neg P$ is true$\rangle$.
AXIOM. The law of **noncontradiction**. For every proposition $P \langle$the proposition $P \wedge \neg P$ is false$\rangle$.
AXIOM. The law of the **excluded middle**. $\forall P \langle P \vee \neg P \rangle$.
AXIOM. The law of **noncontradiction**. $\forall P \langle \neg (P \wedge \neg P) \rangle$.

THEOREM. The law of the **excluded middle** and the law of **noncontradiction** are equivalent.
PROOF.
C0) SHOW: the law of the **excluded middle** is equivalent to the law of **noncontradiction**.
|
| SINCE: by definition, the law of the **excluded middle** is equivalent to $\forall P \langle P \vee \neg P \rangle$,
| SINCE: by definition, the law of **noncontradiction** is equivalent to $\forall P \langle \neg (P \wedge \neg P) \rangle$,
| THEN: by equivalence,
| showing that the law of the **excluded middle** is equivalent to the law of **noncontradiction**
| is equivalent to
| showing that $\forall P \langle P \vee \neg P \rangle$ IFF $\forall P \langle \neg (P \wedge \neg P) \rangle$.
|
| C1) SHOW: $\forall P \langle P \vee \neg P \rangle$ IFF $\forall P \langle \neg (P \wedge \neg P) \rangle$.
| |
| | By $\forall$-hoisting, $\langle \forall P \langle P \vee \neg P \rangle$ IFF $\forall P \langle \neg (P \wedge \neg P) \rangle \rangle$ is equivalent to $\langle \forall P \langle P \vee \neg P$ IFF $\neg (P \wedge \neg P) \rangle \rangle$. (C8)
| |
| | SINCE: by C8), $\langle \forall P \langle P \vee \neg P \rangle$ IFF $\forall P \langle \neg (P \wedge \neg P) \rangle \rangle$ is equivalent to $\langle \forall P \langle P \vee \neg P$ IFF $\neg (P \wedge \neg P) \rangle \rangle$,
| | THEN: by equivalence, showing $\langle \forall P \langle P \vee \neg P \rangle$ IFF $\forall P \langle \neg (P \wedge \neg P) \rangle \rangle$ is equivalent to showing $\langle \forall P \langle P \vee \neg P$ IFF $\neg (P \wedge \neg P) \rangle \rangle$.
| |
| | C2) SHOW: $\langle \forall P \langle P \vee \neg P$ IFF $\neg (P \wedge \neg P) \rangle \rangle$.
| | |
| | | H0) LET: P is a proposition.
| | | C3) SHOW: $P \vee \neg P$ IFF $\neg (P \wedge \neg P)$.
| | | |
| | | | By the duality of conjuction and disjuction, $\neg (P \wedge \neg P)$ IFF $\neg P \vee \neg \neg P$. (C4)
| | | | By NOT-NOT-elimination, $\neg P \vee \neg \neg P$ IFF $\neg P \vee P$. (C5)
| | | | By the commutativity of disjunction, $\neg P \vee P$ IFF $P \vee \neg P$. (C6)
| | | |
| | | | SINCE: by (C4), $\neg (P \wedge \neg P)$ IFF $\neg P \vee \neg \neg P$,
| | | | SINCE: by (C5), $\neg P \vee \neg \neg P$ IFF $\neg P \vee P$,
| | | | SINCE: by (C6), $\neg P \vee P$ IFF $P \vee \neg P$,
| | | | THEN: by the transitivity of equivalence, $\neg (P \wedge \neg P)$ IFF $P \vee \neg P$. (C7)
| | | |
| | | | SINCE: by (C7), $\neg (P \wedge \neg P)$ IFF $P \vee \neg P$,
| | | | THEN: by the commutativity of equivalence, $P \vee \neg P$ IFF $\neg (P \wedge \neg P)$. (C3)
| | | |
| | | SHOWN C3): $P \vee \neg P$ IFF $\neg (P \wedge \neg P)$.
| | |
| | SHOWN C2): $\langle \forall P \langle P \vee \neg P$ IFF $\neg (P \wedge \neg P) \rangle \rangle$.
| |
| SHOWN C1): $\forall P \langle P \vee \neg P \rangle$ IFF $\forall P \langle \neg (P \wedge \neg P) \rangle$.
|
SHOWN C0): the law of the **excluded middle** is equivalent to the law of **noncontradiction**.

DEFINITION. The fundamental abstraction of $\exists$-syntax. The fundamental abstraction of $\forall$-syntax.
Let x be a **variable**.
Let $\varphi[x]$ be an **open sentence** in x (ie. x is a free variable in $\varphi[x]$).
Let X be a set.
0) $\exists x \in X \langle \varphi[x] \rangle$ is DEFINED as $\exists x \langle x \in X$ AND $\varphi[x] \rangle$.
1) $\forall x \in X \langle \varphi[x] \rangle$ is DEFINED as $\forall x \langle$ IF $x \in X$, THEN $\varphi[x] \rangle$.
0') $\exists x \in X \langle \varphi[x] \rangle$ is DEFINED as $\exists x \langle x \in X \wedge \varphi[x] \rangle$.
1') $\forall x \in X \langle \varphi[x] \rangle$ is DEFINED as $\forall x \langle x \in X \Longrightarrow \varphi[x] \rangle$.

DEFINITION. ∃-elimination, aka **existential elimination**, aka existential instantiation.
TODO

DEFINITION. ∀-elimination, aka **universal elimination**, aka universal instantiation.
TODO

The following theorem shows how it's possible to prove an arbitrary statement (is true) assuming an arbitrary false statement (is true). We'll prove that **all math is false**.

THEOREM. All math is false (assuming every lemon is yellow and not every every lemon is yellow).
Let ⟨every lemon is yellow⟩ be a proposition.
Let ⟨all math is false⟩ be a proposition.
0) IF ⟨⟨every lemon is yellow⟩ AND NOT⟨every lemon is yellow⟩⟩,
THEN ⟨all math is false⟩.

PROOF.
H0) LET: ⟨every lemon is yellow ⟩ is a proposition.
H1) LET: ⟨all math is false⟩ is a proposition.
C0) SHOW: IF ⟨⟨every lemon is yellow⟩ AND NOT⟨every lemon is yellow⟩⟩, THEN ⟨all math is false⟩.
|
| H2) LET: ⟨every lemon is yellow⟩.
| H3) LET: NOT⟨every lemon is yellow⟩.
| C1) SHOW: ⟨all math is false⟩.
| |
| | SINCE: by H1), ⟨all math is false⟩ is a proposition,
| | SINCE: by H2), ⟨every lemon is yellow⟩,
| | THEN: by OR-introduction, ⟨all math is false⟩ OR ⟨every lemon is yellow⟩. (C2)
| |
| | SINCE: by H1), ⟨all math is false⟩ is a proposition,
| | SINCE: by H3), NOT⟨every lemon is yellow⟩,
| | THEN: by OR-introduction, ⟨all math is false⟩ OR NOT⟨every lemon is yellow⟩. (C3)
| |
| | SINCE: by C2), ⟨all math is false⟩ OR ⟨every lemon is yellow⟩,
| | SINCE: by C3), ⟨all math is false⟩ OR NOT⟨every lemon is yellow⟩,
| | THEN: by ???, ⟨all math is false⟩. (C1)
| |
| SHOWN: C1) ⟨all math is false⟩.
|
SHOWN: C0) IF ⟨⟨every lemon is yellow⟩ AND NOT⟨every lemon is yellow⟩⟩, THEN ⟨all math is false⟩.

PROOF.
H0) LET: ⟨every lemon is yellow⟩ is a proposition.
H1) LET: ⟨all math is false⟩ is a proposition.
C0) SHOW: IF ⟨⟨every lemon is yellow⟩ AND NOT⟨every lemon is yellow⟩⟩, THEN ⟨all math is false⟩.
|
| H2) LET: ⟨every lemon is yellow⟩ is true.
| H3) LET: NOT⟨every lemon is yellow⟩ is true.
| C1) SHOW: ⟨all math is false⟩ is true.
| |
| | SINCE: by H2), ⟨every lemon is yellow⟩ is true,
| | SINCE: by H1), ⟨all math is false⟩ is a proposition,
| | THEN: by OR-introduction, ⟨every lemon is yellow OR all math is false⟩ is true. (C1)
| |
| | SINCE: by H3), NOT⟨every lemon is yellow⟩ is true,
| | THEN: by negation, ⟨NOT NOT every lemon is yellow⟩ is false. (C2)
| | SINCE: by (C2), ⟨NOT NOT every lemon is yellow⟩ is false,
| | THEN: by NOT-NOT-elimination, ⟨every lemon is yellow⟩ is false. (C3)
| |
| | SINCE: by C1), ⟨every lemon is yellow OR all math is false⟩ is true,
| | SINCE: by C3), ⟨every lemon is yellow⟩ is false,
| | THEN: by OR-elimination, ⟨all math is false⟩ is true. (C1)
| |
| SHOWN: C1) ⟨all math is false⟩ is true.
|
SHOWN: C0) IF ⟨⟨every lemon is yellow⟩ AND NOT⟨every lemon is yellow⟩⟩, THEN ⟨all math is false⟩.

PROOF.
H0) LET: ⟨every lemon is yellow⟩ is a proposition.

H1) Let: ⟨all math is false⟩ is a proposition.
C0) Show: if ⟨⟨every lemon is yellow⟩ and not⟨every lemon is yellow⟩⟩, then ⟨all math is false⟩.
|
| Since: by H0), ⟨every lemon is yellow⟩ is a proposition,
| then: by **the law of noncontradiction**, not ⟨ ⟨every lemon is yellow⟩ and not⟨every lemon is yellow⟩ ⟩. (C1)
|
| Since: by C1), not ⟨ ⟨every lemon is yellow⟩ and not⟨every lemon is yellow⟩ ⟩,
| then: by **the law of false antecedent**, if ⟨⟨every lemon is yellow⟩ and not⟨every lemon is yellow⟩⟩, then ⟨all math is false⟩.
|
Shown: C0) if ⟨⟨every lemon is yellow⟩ and not⟨every lemon is yellow⟩⟩, then ⟨all math is false⟩.

The following is the general version of the previous result. In my opinion, it's easier to read. (So symbols, **properly used**, *can* help understanding).

THEOREM. **The principle of explosion**.
Let P be a proposition.
Let Q be a proposition.
0) if P is true and $\neg P$ is true, then Q is true.

proof.
Let P be a proposition.
Let Q be a proposition.
Let P be true.
Let $\neg P$ be true.
Show Q is true.
|
| Since P and Q are propositions,
| and P is true,
| then, by or-introduction, P or Q is true.
|
| Since $\neg P$ is true,
| then, by negation, $\neg\neg P$ is false,
| Since $\neg\neg P$ is false,
| then, by $\neg\neg$-elimination, P is false.
|
| Since P is false,
| and P or Q is true,
| then, by or-elimination, Q is true.
|
Showed Q is true.

The following is the previous theorem/proof, without explicitly saying *is true*.

THEOREM. **The principle of explosion**.
Let P be a proposition.
Let Q be a proposition.
0) if P and $\neg P$, then Q.

proof.
Let P be a proposition.
Let Q be a proposition.
Let P.
Let $\neg P$.
Show Q.
|
| Since P and Q are propositions,
| and P,
| then, by or-introduction, P or Q.
|
| Since $\neg P$,
| then, by negation, $\neg\neg P$ is false,
| Since $\neg\neg P$ is false,
| then, by $\neg\neg$-elimination, P is false.
|
| Since P is false,
| and P or Q,
| then, by or-elimination, Q.
|
Showed Q.

(Chapter) Sets and functions, a language for mathematics

(Section) Sets

TODO

(Section) Functions

DEFINITION. **Images** and **preimages** of functions.
Let X, Y be sets.
Let $f: X \longrightarrow Y$ be a function.
Let $A \subseteq X$ be a subset of X.
Let $B \subseteq Y$ be a subset of Y.
0) The **image** of $A \subseteq X$, denoted $f_*[A]$, is the set $\{y \in Y \mid \exists a_y \in A \langle\ f: a_y \longmapsto y\ \rangle\}$. (Images **can't be large**.)
1) The **preimage** of $B \subseteq Y$, denoted $f^*[B]$, is the set $\{x \in X \mid \exists b_x \in B \langle\ f: x \longmapsto b_x\ \rangle\}$. (Preimages **can't be small**.)

THEOREM. The fundamental lemma of functions.
Let X, Y be sets.
Let $f: X \longrightarrow Y$ be a function.
0) For every $A \subseteq X$ and $x \in X \langle$ IF $x \in A$, THEN $f[x] \in f_*[A]$ $\rangle$.
1) For every $A \subseteq X$ and $x \in X \langle$ IF $f[x] \in f_*[A]$, MAYBE NOT THEN $x \in A$ $\rangle$.
2) For every $B \subseteq Y$ and $x \in X \langle$ IF $x \in f^*[B]$, THEN $f[x] \in B$ $\rangle$.
3) For every $B \subseteq Y$ and $x \in X \langle$ IF $f[x] \in B$, THEN $x \in f^*[B]$ $\rangle$.
4) For every $A_0, A_1 \subseteq X \langle$ IF $A_0 \subseteq A_1$, THEN $f_*[A_0] \subseteq f_*[A_1]$ $\rangle$. (Images **preserve subsets**.)
5) For every $B_0, B_1 \subseteq Y \langle$ IF $B_0 \subseteq B_1$, THEN $f^*[B_0] \subseteq f^*[B_1]$ $\rangle$. (Preimages **preserve subsets**.)
6) For every $A \subseteq X$ and $B \subseteq Y \langle$ $A \subseteq f^*[B]$ IFF $f_*[A] \subseteq B$ $\rangle$. (**Duality** of images and preimages.)

PROOF of **0)**.
LET X, Y be sets.
LET $f: X \longrightarrow Y$ be a function.
LET $A \subseteq Y$ be a subset of X.
WE SHOW that for every $x \in X \langle$ IF $x \in A$, THEN $f[x] \in f_*[A]$ $\rangle$.
|
| LET $x \in X$ be an element of X.
| LET $x \in A$ be an element of A.
| WE SHOW that $f[x] \in f_*[A]$.
| |
| | By the image definition, $f_*[A]$ is the set $\{y \in Y \mid \exists a_y \in A \langle\ f: a_y \longmapsto y\ \rangle\}$.
| | SINCE WE SHOW that $f[x] \in f_*[A]$,
| | THEN, by the image definition, WE SHOW that there exists $a_y \in A$ so that $f: a_y \longmapsto f[x]$.
| | WE SHOW that there exists $a_y \in A$ so that $f: a_y \longmapsto f[x]$.
| | |
| | | SINCE $x \in A$, AND $f: x \longmapsto f[x]$, THEN there exists $a_y \in A$ so that $f: a_y \longmapsto f[x]$.
| | |
| | THIS SHOWS that there exists $a_y \in A$ so that $f: a_y \longmapsto f[x]$.
| |
| THIS SHOWS that $f[x] \in f_*[A]$.
|
THIS SHOWS that for every $x \in X \langle$ IF $x \in A$, THEN $f[x] \in f_*[A]$ $\rangle$.

PROOF of **1)**.
TODO

PROOF of **2)**.
LET X, Y be sets.
LET $f: X \longrightarrow Y$ be a function.
LET $B \subseteq Y$ be a subset of Y.
WE SHOW that for every $x \in X \langle$ IF $x \in f^*[B]$, THEN $f[x] \in B$ $\rangle$.
|
| LET $x \in X$ be an element of X.
| LET $x \in f^*[B]$ be an element of the preimage $f^*[B]$.
| WE SHOW that $f[x] \in B$.
| |
| | By the preimage definition, $f^*[B]$ is the set $\{x \in X \mid \exists b_x \in B \langle\ f: x \longmapsto b_x\ \rangle\}$.
| | SINCE $f: X \longrightarrow Y$ is a function,
| | THEN, by the function definition, for every $x \in X$ and $y_0, y_1 \in Y \langle$ IF $f: x \longmapsto y_0$ AND $f: x \longmapsto y_1$, THEN $y_0 = y_1$ $\rangle$.
| | SINCE $x \in f^*[B]$, THEN, by the preimage definition, there exists $b_x \in B$ so that $f: x \longmapsto b_x$.
| | SINCE $x \in X$, THEN, by the function definition, there exists $f[x] \in Y$ so that $f: x \longmapsto f[x]$.

| | SINCE $x \in X, b_x, f[x] \in Y$,
| | AND $f : x \longmapsto b_x$,
| | AND $f : x \longmapsto f[x]$,
| | AND for every $x \in X$ and $y_0, y_1 \in Y \langle$ IF $f : x \longmapsto y_0$ AND $f : x \longmapsto y_1$, THEN $y_0 = y_1 \rangle$,
| | THEN, by setting $x \leftarrow x$ and $y_0 \leftarrow b_x$ and $y_1 \leftarrow f[x]$, b_x is equal $f[x]$.
| | SINCE $b_x = f[x]$, AND $b_x \in B$, then, by replacement, $f[x] \in B$.
| |
| THIS SHOWS that $f[x] \in B$. |
THIS SHOWS that for every $x \in X \langle$ IF $x \in f^*[B]$, THEN $f[x] \in B \rangle$.

PROOF of **3)**.
LET X, Y be sets.
LET $f : X \longrightarrow Y$ be a function.
LET $B \subseteq Y$ be a subset of Y.
WE SHOW that for for every $x \in X \langle$ IF $f[x] \in B$, THEN $x \in f^*[B] \rangle$.
|
| LET $x \in X$ be an element of X.
| LET $f[x] \in B$ be an element of B.
| WE SHOW that $x \in f^*[B]$.
| |
| | By the preimage definition, $f^*[B]$ is the set $\{x \in X \mid \exists b_x \in B \langle f : x \longmapsto b_x \rangle\}$.
| | SINCE WE SHOW that $x \in f^*[B]$,
| | THEN, by the preimage definition, WE SHOW that there exists $b_x \in B$ so that $f : x \longmapsto b_x$.
| | WE SHOW that there exists $b_x \in B$ so that $f : x \longmapsto b_x$.
| | |
| | | SINCE $f[x] \in B$, AND $f : x \longmapsto f[x]$, THEN there exists $b_x \in B$ so that $f : x \longmapsto b_x$.
| | |
| | THIS SHOWS that there exists $b_x \in B$ so that $f : x \longmapsto b_x$.
| |
| THIS SHOWS that $x \in f^*[B]$.
|
THIS SHOWS that for for every $x \in X \langle$ IF $f[x] \in B$, THEN $x \in f^*[B] \rangle$.

PROOF of **4)**.
TODO

PROOF of **5)**.
TODO

PROOF of **6)**.
TODO

THEOREM. The fundamental theorem of functions.
Let X, Y be sets.
Let $f : X \longrightarrow Y$ be a function.
0) For every $A \subseteq X \langle f^*[f_*[A]] \supseteq A \rangle$. (**Preimages** of images **can't be small**.)
1) For every $B \subseteq Y \langle f_*[f^*[B]] \subseteq B \rangle$. (**Images** of preimages **can't be large**.)
2) f is **injective** IFF for every $A \subseteq X \langle f^*[f_*[A]] \subseteq A \rangle$. (**Preimages** of injections are **as small as possible**.)
3) f is **surjective** IFF for every $B \subseteq Y \langle f_*[f^*[B]] \supseteq B \rangle$. (**Images** of surjections are **as large as possible**.)

PROOF of **0)**.
LET X, Y be sets.
LET $f : X \longrightarrow Y$ be a function.
LET $A \subseteq X$ be a subset of X.
WE SHOW that $f^*[f_*[A]] \supseteq A$.
|
| By the superset definition, $f^*[f_*[A]] \supseteq A$ is equivalent to $\langle$ for every $x \in A \langle x \in f^*[f_*[A]] \rangle \rangle$.
| SINCE WE SHOW that $f^*[f_*[A]] \supseteq A$,
| AND $f^*[f_*[A]] \supseteq A$ is equivalent to $\langle$ for every $x \in A \langle x \in f^*[f_*[A]] \rangle \rangle$,
| THEN, by replacement, WE SHOW that for every $x \in A \langle x \in f^*[f_*[A]] \rangle$.
| WE SHOW that for every $x \in A \langle x \in f^*[f_*[A]] \rangle$.
| |
| | LET $x \in A$ be an element of A.
| | WE SHOW that $x \in f^*[f_*[A]]$.
| | |
| | | By the preimage definition, $f^*[f_*[A]]$ is the set $\{x \in X \mid \exists b_x \in f_*[A] \langle f : x \longmapsto b_x \rangle\}$.
| | | SINCE WE SHOW that $x \in f^*[f_*[A]]$,
| | | THEN, by the preimage definition, WE SHOW that there exists $b_x \in f_*[A]$ so that $f : x \longmapsto b_x$.

| | | We show that there exists $b_x \in f_*[A]$ so that $f : x \longmapsto b_x$.
| | | |
| | | | By the fundamental abstraction of $\exists$-syntax,
| | | | the sentence $\langle$ there exists $b_x \in f_*[A]$ so that $f : x \longmapsto b_x$ $\rangle$ is equivalent to the sentence $\exists b_x \langle\ b_x \in f_*[A]$ and $f : x \longmapsto b_x\ \rangle$.
| | | | Since We show that $\langle$ there exists $b_x \in f_*[A]$ so that $f : x \longmapsto b_x$ $\rangle$,
| | | | and the sentence $\langle$ there exists $b_x \in f_*[A]$ so that $f : x \longmapsto b_x$ $\rangle$ is equivalent to the sentence $\exists b_x \langle\ b_x \in f_*[A]$ and $f : x \longmapsto b_x\ \rangle$,
| | | | then, by replacement, We show that $\exists b_x \langle\ b_x \in f_*[A]$ and $f : x \longmapsto b_x\ \rangle$.
| | | | We show that $\exists b_x \langle\ b_x \in f_*[A]$ and $f : x \longmapsto b_x\ \rangle$.
| | | | |
| | | | | By the image definition, $f_*[A]$ is the set $\{y \in Y \mid \exists a_y \in A \langle\ f : a_y \longmapsto y\ \rangle\}$.
| | | | | Since must show that $\exists b_x \langle\ b_x \in f_*[A]$ and $f : x \longmapsto b_x\ \rangle$,
| | | | | then, by the image definition, We show that $\exists b_x \langle\ \exists a_y \in A \langle f : a_y \longmapsto b_x \rangle$ and $f : x \longmapsto b_x\ \rangle$.
| | | | | We show that $\exists b_x \langle\ \exists a_y \in A \langle f : a_y \longmapsto b_x \rangle$ and $f : x \longmapsto b_x\ \rangle$.
| | | | | |
| | | | | | By the fundamental abstraction of $\exists$-syntax,
| | | | | | the sentence $\exists a_y \in A \langle f : a_y \longmapsto b_x \rangle$ is equivalent to the sentence $\exists a_y \langle a_y \in A$ and $f : a_y \longmapsto b_x \rangle$.
| | | | | | Since We show that $\exists b_x \langle\ \exists a_y \in A \langle f : a_y \longmapsto b_x \rangle$ and $f : x \longmapsto b_x\ \rangle$,
| | | | | | and the sentence $\exists a_y \in A \langle f : a_y \longmapsto b_x \rangle$ is equivalent to the sentence $\exists a_y \langle a_y \in A$ and $f : a_y \longmapsto b_x \rangle$,
| | | | | | then, by replacement, We show that $\exists b_x \langle\ \exists a_y \langle a_y \in A$ and $f : a_y \longmapsto b_x \rangle$ and $f : x \longmapsto b_x\ \rangle$.
| | | | | | We show that $\exists b_x \langle\ \exists a_y \langle a_y \in A$ and $f : a_y \longmapsto b_x \rangle$ and $f : x \longmapsto b_x\ \rangle$.
| | | | | | |
| | | | | | | Since $x \in A$, and $f : x \longmapsto f[x]$,
| | | | | | | then, by setting $a_y \leftarrow x$ and $b_x \leftarrow f[x]$, we get that $\exists b_x \langle\ \exists a_y \langle a_y \in A$ and $f : a_y \longmapsto b_x \rangle$ and $f : x \longmapsto b_x\ \rangle$.
| | | | | | |
| | | | | | This shows that $\exists b_x \langle\ \exists a_y \langle a_y \in A$ and $f : a_y \longmapsto b_x \rangle$ and $f : x \longmapsto b_x\ \rangle$.
| | | | | |
| | | | | This shows that $\exists b_x \langle\ \exists a_y \in A \langle f : a_y \longmapsto b_x \rangle$ and $f : x \longmapsto b_x\ \rangle$.
| | | | |
| | | | This shows that $\exists b_x \langle\ b_x \in f_*[A]$ and $f : x \longmapsto b_x\ \rangle$.
| | | |
| | | This shows that there exists $b_x \in f_*[A]$ so that $f : x \longmapsto b_x$.
| | |
| | This shows that $x \in f^*[f_*[A]]$.
| |
| This shows that for every $x \in A \langle\ x \in f^*[f_*[A]]\ \rangle$.
|
This shows that $f^*[f_*[A]] \supseteq A$.

proof of **1)**.
TODO

proof of **2)**, only if.
Let X, Y be sets.
Let $f : X \longrightarrow Y$ be a function.
Let $f : X \longrightarrow Y$ be injective.
We show that for every $A \subseteq X \langle\ f^*[f_*[A]] \subseteq A\ \rangle$.
|
| Let $A \subseteq X$ be a subset of X.
| We show that $f^*[f_*[A]] \subseteq A$.
| |
| | By the subset definition, $f^*[f_*[A]] \subseteq A$ is equivalent to $\langle$ for every $x \in f^*[f_*[A]] \langle\ x \in A\ \rangle$ $\rangle$.
| | Since We show that $f^*[f_*[A]] \subseteq A$,
| | and $f^*[f_*[A]] \subseteq A$ is equivalent to $\langle$ for every $x \in f^*[f_*[A]] \langle\ x \in A\ \rangle$ $\rangle$,
| | then, by replacement, We show that for every $x \in f^*[f_*[A]] \langle\ x \in A\ \rangle$.
| | We show that for every $x \in f^*[f_*[A]] \langle\ x \in A\ \rangle$.
| | |
| | | Let $x \in A$ be an element of $f^*[f_*[A]]$.
| | | We show that $x \in A$.
| | | |
| | | | By the preimage definition, $f^*[f_*[A]]$ is the set $\{x \in X \mid \exists b_x \in f_*[A] \langle\ f : x \longmapsto b_x\ \rangle\}$.
| | | | By the image definition, $f_*[A]$ is the set $\{y \in Y \mid \exists a_y \in A \langle\ f : a_y \longmapsto y\ \rangle\}$.
| | | | Since $f : X \longrightarrow Y$ is an injection,
| | | | then, by the injection definition, for every $x_0, x_1 \in X$ and $y \in Y \langle$ if $f : x_0 \longmapsto y$ and $f : x_1 \longmapsto y$, then $x_0 = x_1$ $\rangle$.
| | | | Since x is in $f^*[f_*[A]]$, then, by the preimage definition, there exists $b_x \in f_*[A]$ so that $f : x \longmapsto b_x$.
| | | | Since b_x is in $f_*[A]$, then, by the image definition, there exists $a_y \in A$ so that $f : a_y \longmapsto b_x$.
| | | | Since $x, a_y \in X$, and $b_x \in Y$
| | | | and $f : x \longmapsto b_x$,

| | | | AND $f : a_y \longmapsto b_x$,
| | | | AND for every $x_0, x_1 \in X$ and $y \in Y \langle$ IF $f : x_0 \longmapsto y$ AND $f : x_1 \longmapsto y$, THEN $x_0 = x_1 \rangle$,
| | | | THEN, by setting $x_0 \leftarrow x$ and $x_1 \leftarrow a_y$ and $y \leftarrow b_x$, x is equal to a_y.
| | | | SINCE $x = a_y$, AND $a_y \in A$ THEN, by replacement, $x \in A$.
| | | |
| | | THIS SHOWS that $x \in A$.
| | |
| | THIS SHOWS that for every $x \in f^*[f_*[A]] \langle\ x \in A\ \rangle$.
| |
| THIS SHOWS that $f^*[f_*[A]] \subseteq A$.
|
THIS SHOWS that for every $A \subseteq X \langle\ f^*[f_*[A]] \subseteq A\ \rangle$.

PROOF of **2)**, if.
LET X, Y be sets.
LET $f : X \longrightarrow Y$ be a function.
LET $f : X \longrightarrow Y$ satisfy $\langle$ for every $A \subseteq X \langle\ f^*[f_*[A]] \subseteq A\ \rangle\ \rangle$.
WE SHOW that $f : X \longrightarrow Y$ is injective.
|
| SINCE WE SHOW that f is injective,
| THEN, by the injective definition, WE SHOW that for every $x_0, x_1 \in X$ and $y \in Y \langle$ IF $f : x_0 \longmapsto y$ AND $f : x_1 \longmapsto y$, THEN $x_0 = x_1 \rangle$.
| WE SHOW that for every $x_0, x_1 \in X$ and $y \in Y \langle$ IF $f : x_0 \longmapsto y$ AND $f : x_1 \longmapsto y$, THEN $x_0 = x_1 \rangle$.
| |
| | LET $x_0, x_1 \in X$.
| | LET $y \in Y$.
| | WE SHOW that IF $f : x_0 \longmapsto y$ AND $f : x_1 \longmapsto y$, THEN $x_0 = x_1$.
| | |
| | | LET $f : x_0 \longmapsto y$ AND $f : x_1 \longmapsto y$.
| | | WE SHOW that $x_0 = x_1$.
| | | |
| | | | TODO
| | | |
| | | THIS SHOWS that $x_0 = x_1$.
| | |
| | THIS SHOWS that IF $f : x_0 \longmapsto y$ AND $f : x_1 \longmapsto y$, THEN $x_0 = x_1$.
| |
| THIS SHOWS that for every $x_0, x_1 \in X$ and $y \in Y \langle$ IF $f : x_0 \longmapsto y$ AND $f : x_1 \longmapsto y$, THEN $x_0 = x_1 \rangle$.

THIS SHOWS that $f : X \longrightarrow Y$ is injective.

PROOF of **3)**, only if.
TODO

PROOF of **3)**, if.
TODO

THEOREM. The fundamental meta-theorem of equations.
Let A be a "math expression".
Let B be a "math expression".
Let f be a function from "math expressions" to "math expressions" (ie. the image under f of each "math expression" is unique).
0) IF A **equals** B, THEN $f[A]$ **equals** $f[B]$.
PROOF. I don't know.

(Chapter) Convergence, pillar of analysis

Limits are the workhorse of **analysis**. In analysis, "everything is a limit". Or something.

Derivatives are limits. **Integrals** are limits. **Continuity** is defined using limits. Even **equality** can be defined using limits (kinda).

But we'll prefer a different language: that of **convergence**. Limits and convergence are the same idea.
You can say that analysis is built on **limits**.
You can say that analysis is built on **convergence**.

DEFINITION. **Limits** and **convergence** of sequences.
Let $f: \mathbf{N} \longrightarrow \mathbf{R}$ be a sequence.
Let $L \in \mathbf{R}$ be a real number in the codomain of f.

0) f has **limit** L (at *infinity*), denoted $f \longrightarrow L$, IFF
for every precision $\epsilon \in \mathbf{R}^+$
there exists a threshold $N_\epsilon \in \mathbf{N}$ so that
for every $x \in \mathbf{N}$ in the domain $\langle$
IF x is in the ∞-ball $(N_\epsilon..\infty)_{\mathbf{N}}$, THEN $f[x]$ is in the ϵ-ball $(L-\epsilon..L+\epsilon)_{\mathbf{R}}$
$\rangle$.

1) f **converges** to L (at *infinity*), denoted $f \longrightarrow L$, IFF
for every precision $\epsilon \in \mathbf{R}^+$
there exists a threshold $N_\epsilon \in \mathbf{N}$ so that
for every $x \in \mathbf{N}$ in the domain $\langle$
IF x is in the ∞-ball $(N_\epsilon..\infty)_{\mathbf{N}}$, THEN $f[x]$ is in the ϵ-ball $(L-\epsilon..L+\epsilon)_{\mathbf{R}}$
$\rangle$.

2) f has **limit** L (at *infinity*) IFF f **converges** to L (at *infinity*).

DEFINITION. **Limits** and **convergence** of functions.
Let $f: A \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be function.
Let $a \in A$ be a real number in the domain of f.
Let $L \in \mathbf{R}$ be a real number in the codomain of f.

0) f has **limit** L at a, denoted $f \longrightarrow L@a$, IFF
for every precision $\epsilon \in \mathbf{R}^+$
there exists a threshold $\delta_\epsilon \in \mathbf{R}^+$ so that
for every $x \in A$ in the domain $\langle$
IF x is in the δ-ball $(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}}$, THEN $f[x]$ is in the ϵ-ball $(L-\epsilon..L+\epsilon)_{\mathbf{R}}$
$\rangle$.

1) f **converges** to L at a, denoted $f \longrightarrow L@a$, IFF
for every precision $\epsilon \in \mathbf{R}^+$
there exists a threshold $\delta_\epsilon \in \mathbf{R}^+$ so that
for every $x \in A$ in the domain $\langle$
IF x is in the δ-ball $(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}}$, THEN $f[x]$ is in the ϵ-ball $(L-\epsilon..L+\epsilon)_{\mathbf{R}}$
$\rangle$.

2) f has **limit** L at a IFF f **converges** to L at a.

(Section) Open sets, a language for convergence

THEOREM. **Convergence** via open sets.
Let $f: A \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be function.
Let $a \in A$ be a real number in the domain of f.
Let $L \in \mathbf{R}$ be a real number in the codomain of f.

0) f **converges** to L at a IFF for every $\epsilon \in \mathbf{R}^+$, there exists $\delta_\epsilon \in \mathbf{R}^+$ so that, for every $x \in A$ in the domain $\langle$
IF $x \in (a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}}$, THEN $f[x] \in (L-\epsilon..L+\epsilon)_{\mathbf{R}}$
$\rangle$.

1) f **converges** to L at a IFF for every $\epsilon \in \mathbf{R}^+$, there exists $\delta_\epsilon \in \mathbf{R}^+$ so that, for every $x \in A$ in the domain $\langle$
IF $x \in (a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}}$, THEN $x \in f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]$
$\rangle$.

2) f **converges** to L at a IFF for every $\epsilon \in \mathbf{R}^+$, there exists $\delta_\epsilon \in \mathbf{R}^+ \langle$
$(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}} \subseteq f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]$
$\rangle$.

3) f **converges** to L at a IFF for every $\epsilon \in \mathbf{R}^+$, there exists $\delta_\epsilon \in \mathbf{R}^+ \langle$
$f_*[(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}}] \subseteq (L-\epsilon..L+\epsilon)_{\mathbf{R}}$
$\rangle$.

4) f **converges** to L at a IFF for every open ball $B[L,\epsilon]$ at L, there exists an open ball $B[a,\delta_\epsilon]$ at $a \langle$
$B[a,\delta_\epsilon] \subseteq f^*[B[L,\epsilon]]$
$\rangle$.

5) f **converges** to L at a IFF for every open ball $B[L,\epsilon]$ at L, there exists an open ball $B[a,\delta_\epsilon]$ at $a \langle$
$f_*[B[a,\delta_\epsilon]] \subseteq B[L,\epsilon]$

$\rangle$.

6) f **converges** to L at a IFF for every open ball $B[L,\epsilon]$ at $L\langle$
$f^*[B[L,\epsilon]]$ is open
$\rangle$.

PROOF of **0)**.
This is just the convergence definition, for reference =)

PROOF of **1)**, only if.
Let $f : A\subseteq\mathbf{R}\longrightarrow\mathbf{R}$ be function.
Let $a\in A$ be a real number in the domain of f.
Let $L\in\mathbf{R}$ be a real number in the codomain of f.
Let f **converge** to L at a.
WE SHOW that for every $\epsilon\in\mathbf{R}^+$, there exists $\delta_\epsilon\in\mathbf{R}^+$ so that, for every $x\in A\langle$ IF $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}$, THEN $x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$.

By the fundamental lemma of functions, for every subset $B\subseteq\mathbf{Cod}[f]$, for every $x\in\mathbf{Dom}[f]\langle$ $f[x]\in B$ IFF $x\in f^*[B]$ $\rangle$.
SINCE $(L-\epsilon..L+\epsilon)_\mathbf{R}$ is a subset of $\mathbf{Cod}[f]$, AND x is in $\mathbf{Dom}[f]$,
THEN, by the fundamental lemma of functions and setting $B\leftarrow(L-\epsilon..L+\epsilon)_\mathbf{R}$, we get that $f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ IFF $x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$.

SINCE f **converges** to L at a,
THEN, by the convergence definition,
for every $\epsilon\in\mathbf{R}^+$, there exists $\delta_\epsilon\in\mathbf{R}^+$ so that, for every $x\in A\langle$ IF $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{N}$, THEN $f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ $\rangle$.

SINCE $f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ IFF $x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$,
AND for every $\epsilon\in\mathbf{R}^+$, there exists $\delta_\epsilon\in\mathbf{R}^+$ so that, for every $x\in A\langle$ IF $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{N}$, THEN $f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ $\rangle$,
THEN, by replacing $f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ with $x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$,
for every $\epsilon\in\mathbf{R}^+$, there exists $\delta_\epsilon\in\mathbf{R}^+$ so that, for every $x\in A\langle$ IF $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}$, THEN $x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$.

THIS SHOWS that for every $\epsilon\in\mathbf{R}^+$, there exists $\delta_\epsilon\in\mathbf{R}^+$ so that, for every $x\in A\langle$ IF $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}$, THEN $x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$.

PROOF of **1)**, if.
Let $f : A\subseteq\mathbf{R}\longrightarrow\mathbf{R}$ be function.
Let $a\in A$ be a real number in the domain of f.
Let $L\in\mathbf{R}$ be a real number in the codomain of f.
Let $\langle$ for every $\epsilon\in\mathbf{R}^+$, there exists $\delta_\epsilon\in\mathbf{R}^+$ so that, for every $x\in A\langle$ IF $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}$, THEN $x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$ $\rangle$.
WE SHOW that f **converges** to L at a.

TODO

THIS SHOWS that f **converges** to L at a.

PROOF of **1)**, direct.
Let $f : A\subseteq\mathbf{R}\longrightarrow\mathbf{R}$ be function.
Let $a\in A$ be a real number in the domain of f.
Let $L\in\mathbf{R}$ be a real number in the codomain of f.
WE SHOW that $\langle$ $f\longrightarrow L@a$ $\rangle$ IFF $\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$ $\rangle$.

By the convergence definition,
$\langle$ $f\longrightarrow L@a$ $\rangle$ is equivalent to $\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ $\rangle$ $\rangle$.
SINCE $\langle$ $f\longrightarrow L@a$ $\rangle$ is equivalent to $\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ $\rangle$ $\rangle$,
AND WE SHOW that $\langle$ $f\longrightarrow L@a$ $\rangle$ IFF $\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$ $\rangle$,
THEN, by replacement, WE SHOW that
$\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ $\rangle$ $\rangle$
IFF
$\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$ $\rangle$.
WE SHOW that
$\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ $\rangle$ $\rangle$
IFF
$\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$ $\rangle$.

SINCE WE SHOW that
$\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$ $\rangle$ $\rangle$
IFF
$\langle$ $\forall\epsilon\in\mathbf{R}^+$ $\exists\delta_\epsilon\in\mathbf{R}^+$ $\forall x\in A\langle$ $x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$ $\rangle$ $\rangle$.
THEN, by the fundamental lemma of first-order classical logic, we can peel the outer quantifier layers that are equal, so
WE SHOW that
$x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow f[x]\in(L-\epsilon..L+\epsilon)_\mathbf{R}$
IFF
$x\in(a-\delta_\epsilon..a+\delta_\epsilon)_\mathbf{R}\Longrightarrow x\in f^*[(L-\epsilon..L+\epsilon)_\mathbf{R}]$.

We show that
$x{\in}(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}} \implies f[x]{\in}(L-\epsilon..L+\epsilon)_{\mathbf{R}}$
IFF
$x{\in}(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}} \implies x{\in}f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]$.

Let $x{\in}(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}}$.
We show that
$f[x]{\in}(L-\epsilon..L+\epsilon)_{\mathbf{R}}$
IFF
$x{\in}f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]$.

By the fundamental lemma of functions, setting B to $(L-\epsilon..L+\epsilon)_{\mathbf{R}}$, we get that
$f[x]{\in}B$ IFF $x{\in}f^*[B]$, meaning
$f[x]{\in}(L-\epsilon..L+\epsilon)_{\mathbf{R}}$ IFF $x{\in}f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]$.

This shows that
$f[x]{\in}(L-\epsilon..L+\epsilon)_{\mathbf{R}}$
IFF
$x{\in}f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]$.

This shows that
$x{\in}(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}} \implies f[x]{\in}(L-\epsilon..L+\epsilon)_{\mathbf{R}}$
IFF
$x{\in}(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}} \implies x{\in}f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]$.

This shows that
$\langle\ \forall\epsilon{\in}\mathbf{R}^+\ \exists\delta_\epsilon{\in}\mathbf{R}^+\ \forall x{\in}A\langle\ x{\in}(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}} \implies f[x]{\in}(L-\epsilon..L+\epsilon)_{\mathbf{R}}\ \rangle\ \rangle$
IFF
$\langle\ \forall\epsilon{\in}\mathbf{R}^+\ \exists\delta_\epsilon{\in}\mathbf{R}^+\ \forall x{\in}A\langle\ x{\in}(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}} \implies x{\in}f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]\ \rangle\ \rangle$.

This shows that $\langle\ f{\longrightarrow}L@a\ \rangle$ IFF $\langle\ \forall\epsilon{\in}\mathbf{R}^+\ \exists\delta_\epsilon{\in}\mathbf{R}^+\ \forall x{\in}A\langle\ x{\in}(a-\delta_\epsilon..a+\delta_\epsilon)_{\mathbf{R}} \implies x{\in}f^*[(L-\epsilon..L+\epsilon)_{\mathbf{R}}]\ \rangle\ \rangle$.

PROOF of **2)**.
Let $f : A{\subseteq}\mathbf{R}{\longrightarrow}\mathbf{R}$ be function.
Let $a{\in}A$ be a real number in the domain of f.
Let $L{\in}\mathbf{R}$ be a real number in the codomain of f.
TODO

PROOF of **3)**.
Let $f : A{\subseteq}\mathbf{R}{\longrightarrow}\mathbf{R}$ be function.
Let $a{\in}A$ be a real number in the domain of f.
Let $L{\in}\mathbf{R}$ be a real number in the codomain of f.
TODO

PROOF of **4)**.
Let $f : A{\subseteq}\mathbf{R}{\longrightarrow}\mathbf{R}$ be function.
Let $a{\in}A$ be a real number in the domain of f.
Let $L{\in}\mathbf{R}$ be a real number in the codomain of f.
TODO

PROOF of **5)**.
Let $f : A{\subseteq}\mathbf{R}{\longrightarrow}\mathbf{R}$ be function.
Let $a{\in}A$ be a real number in the domain of f.
Let $L{\in}\mathbf{R}$ be a real number in the codomain of f.
TODO

PROOF of **6)**.
Let $f : A{\subseteq}\mathbf{R}{\longrightarrow}\mathbf{R}$ be function.
Let $a{\in}A$ be a real number in the domain of f.
Let $L{\in}\mathbf{R}$ be a real number in the codomain of f.
TODO

(Section) The fundamental theorem of ϵ-equality

DEFINITION. Let $a, b{\in}\mathbf{R}$ be real numbers.
0) a is **under** b IFF $a{<}b$.
1) a is **over** b IFF $a{>}b$.
2) a is **at most** b IFF $a{\leq}b$.
3) a is **at least** b IFF $a{\geq}b$.

LEMMA. Let $a,b\in\mathbf{R}$ be real numbers.

0) IF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$, THEN $|a-b|\leq 0$.

1) IF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$, THEN $|a-b|\notin\mathbf{R}^+$.

2) IF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|-\epsilon\in\mathbf{R}^-$, THEN $|a-b|\notin\mathbf{R}^+$.

THEOREM. The fundamental theorem of ϵ-equality, aka the fundamental theorem of analytic equality.
Let $a,b\in\mathbf{R}$ be real numbers.

0) a **equals** b IFF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$.

In symbols,

for every $a,b\in\mathbf{R}\langle$
$a=b$ IFF for every $\epsilon\in\mathbf{R}^+\langle$
$|a-b|<\epsilon$
$\rangle$
$\rangle$.

PROOF.
H0) LET $a,b\in\mathbf{R}$ be real numbers.
WE SHOW that a **equals** b IFF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$.
C0) By the absolute value definition, $|0|=0$.

WE SHOW that IF a **equals** b, THEN for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$.
H1) LET a **equal** b.
H2) LET $\epsilon\in\mathbf{R}^+$.
WE SHOW that $|a-b|<\epsilon$.
SINCE, by H1), $a=b$, THEN, by the existence of additive inverses for reals, C1) $a-b=0$.
SINCE, by C1), $a-b=0$, THEN, by the fundamental meta-theorem of equations, C2) $|a-b|=|0|$.
SINCE, by C2), $|a-b|=|0|$, AND, by C0) $|0|=0$, THEN, by replacement, C3) $|a-b|=0$.
SINCE, by the $\mathbf{R}$ axioms, 0 is under every positive real, AND, by H2), ϵ is positive, THEN, by replacement, C4) 0 is under ϵ.
SINCE, by C3), $|a-b|=0$, AND, by C4), $0<\epsilon$, THEN, by replacement, $|a-b|<\epsilon$.
THIS SHOWS that $|a-b|<\epsilon$.
THIS SHOWS that C5) IF a **equals** b, THEN for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$.

WE SHOW that IF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$, THEN a **equals** b.
H3) LET $\epsilon\in\mathbf{R}^+$.
H4) LET $|a-b|<\epsilon$.
H5) LET a **not equal** b, for CONTRADICTION.
We must find a contradiction.
SINCE, by H5), $a\neq b$, THEN, by the existence of additive inverses for reals, C6) $a-b\neq 0$.
SINCE, by C6), $a-b\neq 0$, THEN, by the fundamental meta-theorem of equations, C7) $|a-b|\neq|0|$.
SINCE, by C7), $|a-b|\neq|0|$, AND, by C0), $|0|=0$, THEN, by replacement, C8) $|a-b|\neq 0$.
SINCE, by C8), $|a-b|\neq 0$, THEN, by the trichomotoy of reals, C9) $|a-b|<0$ or $|a-b|>0$.
SINCE, by C9), $|a-b|<0$ or $|a-b|>0$, AND absolute values are always nonnegative, THEN by $\vee$-elimination, C10) $|a-b|>0$.
SINCE, by C10), $|a-b|>0$, THEN, by the positive reals definition $\mathbf{R}^+$, C11) $|a-b|\in\mathbf{R}^+$.
SINCE, by H3) and H4), for every $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$, THEN, by the previous lemma, C12) $|a-b|\notin\mathbf{R}^+$.
SINCE, by C11), $|a-b|\in\mathbf{R}^+$, AND, by C12), $|a-b|\notin\mathbf{R}^+$, THEN there's a CONTRADICTION.
THIS SHOWS that a **equals** b, by the law of non-contradiction.
THIS SHOWS that C13) IF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$, THEN a **equals** b.

SINCE, by C5), IF a **equals** b, THEN for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$,
AND, by C13), IF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$, THEN a **equals** b,
THEN, by the IFF definition, a **equals** b IFF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$.

THIS SHOWS that a **equals** b IFF for every positive $\epsilon\in\mathbf{R}^+$ it's true that $|a-b|<\epsilon$.

THEOREM. The triangle inequality for $\mathbf{R}$.
Let $a,b\in\mathbf{R}$ be real numbers.

0) $|a+b|$ is at most $|a|+|b|$.

In symbols,

for every $a,b\in\mathbf{R}\langle$
$|a+b|\leq|a|+|b|$
$\rangle$.

PROOF. TODO

(Chapter) The three fundamental theorems of calculus

THEOREM. The **first fundamental lemma of calculus**, aka **the mean value theorem for derivatives**, aka the **local-to-global principle** of differential calculus.

THEOREM. The **second fundamental lemma of calculus**, aka **the mean value theorem for integrals**, aka the **local-to-global principle** of integral calculus.

THEOREM. The **first fundamental theorem of calculus**, aka the differential of the **area function of a function** is the differential of the **function**.

THEOREM. The **second fundamental theorem of calculus**, (high-dimensional) integration on a (high-dimensional) interior is (low-dimensional) integration on a (low-dimensional) boundary.

THEOREM. The **third fundamental theorem of calculus**, aka **Taylor's differential expansion**, aka Taylor's analytic approximation, aka Taylor's theorem.

By the **First Fundamental Theorem of Calculus**, if a function is **Riemann integrable** and **continuous**, then it has an **antiderivative**. Also, the antiderivative is **continuous**.

More specifically, by the First Fundamental Theorem of Calculus, if a function f is Riemann integrable and continuous, then it has an antiderivative F, and the antiderative is precisely the (continuous) function $F : x \longmapsto \int_{[a..x]} f$.

(Chapter) Topology

Topology is the study of **continuous functions**.
To talk about continuous functions, we must talk about **open sets**.
Open sets are *not* defined *directly*, but indirectly in terms of their set-theoretic *behavior*: how they behave under **unions** and **intersections**.
So, I can never tell you what an open set *is*, only *how it behaves*. It's its *behavior* that defines it.

THEOREM. The fundamental duality of **open topologies** and **closed topologies**.

LEMMA. The fundamental lemma of continuity and compacteness.
Images of continuous functions on compact sets are compact.
If the domain of a continuous function is compact, then its image is compact.

LEMMA.
Let X be a totally-ordered topological space.
0) IF X has no min, THEN the 2-set of ∞-balls $\{B{\subseteq}X \mid \exists a{\in}X\langle\ B{=}(a..{+}\infty)\ \rangle\}$ is an open cover of X.
1) IF X has no max, THEN the 2-set of ∞-balls $\{B{\subseteq}X \mid \exists a{\in}X\langle\ B{=}({-}\infty..a)\ \rangle\}$ is an open cover of X.
2) IF X has min m, THEN the 2-set of ∞-balls $\{B{\subseteq}X \mid \exists a{\in}X\langle\ B{=}(a..{+}\infty)\ \rangle\}$ is an open cover of $X{-}\{m\}$.
3) IF X has max M, THEN the 2-set of ∞-balls $\{B{\subseteq}X \mid \exists a{\in}X\langle\ B{=}({-}\infty..a)\ \rangle\}$ is an open cover of $X{-}\{M\}$.
PROOF of **1)**.
LET X be a totally-ordered topological space.
LET X have no max.
LET $\mathcal{B}$ be the 2-set of ∞-balls $\{B{\subseteq}X \mid \exists a{\in}X\langle\ B{=}({-}\infty..a)\ \rangle\}$.
WE SHOW that $\mathcal{B}$ is an open cover of X.
SINCE WE SHOW that $\mathcal{B}$ is an open cover of X, THEN, by the open cover definition, WE SHOW that X is a subset of $\cup\mathcal{B}$.
WE SHOW that x is an element of $\cup\mathcal{B}$.
LET x **not** be an element of $\cup\mathcal{B}$, for CONTRADICTION.
SINCE x is **not** in $\cup\mathcal{B}$, THEN, by negating the union definition, there doesn't exist $B{\in}\mathcal{B}$ so that $x{\in}B$.
SINCE $\neg\exists B{\in}\mathcal{B}\langle\ x{\in}B\ \rangle$, THEN, by the rules of classical logic, $\forall B{\in}\mathcal{B}\langle\ x{\notin}B\ \rangle$.

SINCE X has no max, THEN, by negating the max definition, there doesn't exist $M{\in}X$ so that for all $y{\in}X$ it's true that $y{\leq}M$.
SINCE $\neg\exists M{\in}X\ \forall y{\in}X\langle\ y{\leq}M\ \rangle$, THEN, by the rules of classical logic, $\forall M{\in}X\ \exists y{\in}X\langle\ y{>}M\ \rangle$.
SINCE $\forall M{\in}X\ \exists y{\in}X\langle\ y{>}M\ \rangle$, AND $x{\in}X$, THEN, by plugging $M{:=}x$, there exists $x'{\in}X$ so that $x'{>}x$.
SINCE $x{<}x'$, AND $x{\in}X$, AND $x'{\in}X$, THEN, by the ball definition, x is in the ball $({-}\infty..x')$.
SINCE $x'{\in}X$, THEN, by the $\mathcal{B}$ definition, the ball $({-}\infty..x')$ is in $\mathcal{B}$.
SINCE $({-}\infty..x'){\in}\mathcal{B}$, AND $x{\in}({-}\infty..x')$, THEN there exists $B{\in}\mathcal{B}$ so that $x{\in}B$.

SINCE $\forall B{\in}\mathcal{B}\langle\ x{\notin}B\ \rangle$, AND $\exists B{\in}\mathcal{B}\langle\ x{\in}B\ \rangle$, THEN there's a CONTRADICTION.
THIS SHOWS that x is an element of $\cup\mathcal{B}$, by the law of non-contradiction.
THIS SHOWS that X is a subset of $\cup\mathcal{B}$.
THIS SHOWS that $\cup\mathcal{B}$ is an open cover of X.

THEOREM. The extreme value theorem for topological spaces.
Let X be a **compact** topological space.
Let Y be a **totally-ordered** topological space.
Let $f:X{\longrightarrow}Y$ be continuous.
0) There exist $a,b{\in}X$ so that for every $x{\in}X$ it's true that $f[x]{\in}[f[a]..f[b]]$.
The point $f[a]{\in}X$ is called the **min** of f.
The point $f[b]{\in}X$ is called the **max** of f.
The point $a{\in}X$ is called the **argmin** of f.
The point $b{\in}X$ is called the **argmax** of f.
PROOF.
LET X be a **compact** topological space.
LET Y be a **totally-ordered** topological space.
LET $f:X{\longrightarrow}Y$ be continuous.
WE SHOW that there exist $a,b{\in}X$ so that for every $x{\in}X$ it's true that $f[x]{\in}[f[a]..f[b]]$.

SINCE X is compact AND f is continuous, THEN, by the fundamental lemma of continuity and compactness, the image $f_*[X]$ is compact.

LET m be the min of $f_*[X]$. (Why does this exist? This is what we want to proof!)
LET M be the max of $f_*[X]$. (Why does this exist? This is what we want to proof!)
SINCE m is the min of $f_*[X]$, THEN, by the min definition, m is in $f_*[X]$.
SINCE M is the max of $f_*[X]$, THEN, by the max definition, M is in $f_*[X]$.
SINCE $m{\in}f_*[X]$, THEN, by the $f_*[X]$ definition, there exists $a{\in}X$ so that $f:a{\longmapsto}m$.
SINCE $M{\in}f_*[X]$, THEN, by the $f_*[X]$ definition, there exists $a{\in}X$ so that $f:b{\longmapsto}M$.

LET $f_*[X]$ have no max, for CONTRADICTION.
LET $\mathcal{B}$ be the 2-set of ∞-balls $\{B{\subseteq}f_*[X] \mid \exists y{\in}f_*[X]\langle\ B{=}({-}\infty..y)\ \rangle\}$.
SINCE the domain of X, AND the codomain of f is Y, THEN by the image definition, the image $f_*[X]$ is a subset of Y.

SINCE $f_*[X]$ is a subset of Y, and Y is totally-ordered, THEN, by XX, $f_*[X]$ is totally ordered.
SINCE $f_*[X]$ has no max, AND $f_*[X]$ is totally-ordered, THEN, by lemma XX, the 2-set $\mathcal{B}$ is an open cover of $f_*[X]$.
SINCE the 2-set $\mathcal{B}$ is an open cover of $f_*[X]$, AND $f_*[X]$ is compact,
THEN, by the compactness definition, it has a finite subcover $\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$.
SINCE the cover $\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$ is finite, THEN the set $\{y_0, y_1, \ldots, y_n\}$ of boundary points is finite.
SINCE the set $\{y_0, y_1, \ldots, y_n\}$ is finite, THEN, by XX, it has a maximum M.
SINCE M is the max of $\{y_0, y_1, \ldots, y_n\}$, THEN, by the max definition, M is an element of $\{y_0, y_1, \ldots, y_n\}$.
SINCE M is an element of $\{y_0, y_1, \ldots, y_n\}$, AND $\{y_0, y_1, \ldots, y_n\}$ is a subset of $f_*[X]$,
THEN by the properties of subsets, M is an element of $f_*[X]$.
SINCE M is an element of $f_*[X]$, AND $\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$ covers $f_*[X]$,
THEN, by the cover definition, M is an element of the union $\cup\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$.
SINCE M is an element of the union $\cup\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$,
THEN, by the union definition, there exists $(-\infty..y_i) \in \{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$ so that $M \in (-\infty..y_i)$.
SINCE M is an element of $(-\infty..y_i)$, AND $(-\infty..y_i)$ in an element of $\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$,
THEN $M \in (-\infty..y_0)$ or $M \in (-\infty..y_1)$ or $\ldots$ $M \in (-\infty..y_n)$.

SINCE M is an element of $\{y_0, y_1, \ldots, y_n\}$,
AND every element of $\{y_0, y_1, \ldots, y_n\}$ is a boundary point of an element of $\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$,
THEN M is a boundary point of an element of $\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$.
SINCE M is a boundary point of an element of $\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$,
AND every element of $\{(-\infty..y_0), (-\infty..y_1), \ldots, (-\infty..y_n)\}$ is an open ball,
AND open balls don't contain boundary points,
THEN $M \notin (-\infty..y_0)$ or $M \notin (-\infty..y_1)$ or $\ldots$ $M \notin (-\infty..y_n)$.

SINCE $M \notin (-\infty..y_0)$ or $M \notin (-\infty..y_1)$ or $\ldots$ $M \notin (-\infty..y_n)$,
AND $M \in (-\infty..y_0)$ or $M \in (-\infty..y_1)$ or $\ldots$ $M \in (-\infty..y_n)$,
THEN there's a CONTRADICTION.

(subsection) **[...]**

[...]

A **category** is **dots**, **arrows** (between **dots**), and **gluing conditions** (between **arrows**).
The **dots** and **arrows** *can* be explicitly visualized (they're concrete *things*).
The **gluing conditions** *can't* be explicitly visualized (they're abstract *meta-things*, or something).

EXAMPLE. **The two-equals-one axiom**.
I used to think that the arrows
$f\ : X \longrightarrow Y$
$g\ : Y \longrightarrow Z$
$h\ : X \longrightarrow Z$
$1_X : X \longrightarrow X$
$1_Y : Y \longrightarrow Y$
$1_Z : Z \longrightarrow Z$
formed a category. But they don't. **Dots** and **arrows** alone don't make a category. We need **gluing conditions**, too.

Trick question: *how many* **arrows** does this category have?
I used to think it had 6: $f, g, h, 1_X, 1_Y, 1_Z$. But it doesn't.
It has 7 arrows: SINCE the target of f EQUALS the source of g, THEN, by the category axioms, there exists a arrow gf.
So our collection of arrows grows by 1: $f, g, h, 1_X, 1_Y, 1_Z, gf$.
Or does it?
Notice that the target of 1_X equals the source of f, so we also get the arrow $f1_X$.
For analogous reasons, we also get the arrows $1_Y f, g1_Y, 1_Z g, h1_X, 1_Z h$.
So our collection of arrows grows to: $f, g, h, 1_X, 1_Y, 1_Z, gf, f1_X, 1_Y f, g1_Y, 1_Z g, h1_X, 1_Z h$.
Or does it?
The collection of arrows $f, g, h, 1_X, 1_Y, 1_Z, gf, f1_X, 1_Y f, g1_Y, 1_Z g, h1_X, 1_Z h$ on its own doesn't form a category: it's missing **gluing conditions**. And we can't just go about choosing any old **gluing conditions** that we please; nope. Our **gluing conditions** must satisfy the **category axioms**. The following set of gluing conditions does the trick:
$gf \ = \ h$
$f1_X \ = \ f$
$1_Y f \ = \ f$
$g1_Y \ = \ g$
$1_Z g \ = \ g$
$h1_X \ = \ h$
$1_Z h \ = \ h$
Aha! So under these **gluing conditions**, the arrow gf "*equals*" the arrow h (whatever "*equals*" means), and similarly for other arrows.
This means that our collection of 6+7 arrows
$f, g, h, 1_X, 1_Y, 1_Z, gf, f1_X, 1_Y f, g1_Y, 1_Z g, h1_X, 1_Z h$
"*collapses down*" to the original 6 arrows
$f, g, h, 1_X, 1_Y, 1_Z$.

Objects and **morphisms** can be *visualized* as **dots** and **arrows**.
But how do we *visualize* the fact that (for instance) $gf{=}h$?
I don't know, and I suspect we can't (it's a *meta-thing*...), because gf is the composition of f with g (so gf is a path of length 2), but h is a single arrow (it's a path of length 1)!
How *can* the *two* arrows f and g *equal* the *one* arrow h? I don't know. It's just an axiom for this category. And I don't know how to visualize it. But I think of it as the axiom 2=1: **two arrows equal one arrow**.

So, for this collection of arrows, under these gluing conditions, the arrows f, g, h satisfy the 2=1 axiom. (And other arrows do as well.)

When thinking about **categories**:
we *try* to "forget" about the *internal structure* of **objects**, and think of objects as *structureless point-particles*,
we *try* to "forget" about the **objects** altogether, and think only in terms of the **arrows**.

Categories are **posets** in the next dimension.
∞-groupoids are **sets** in the next dimension.

DEFINITION. **Categories**. The category axioms.
A **category** $\mathcal{C}$ satisfies the following sentences.
0) *Existence of arrows*:
there exists a class $\mathbf{Hom}[\mathcal{C}]$ of $\mathcal{C}$-**arrows**.
1) *Existence of source-arrows and target-arrows*:
for every $\mathcal{C}$-arrow $f{\in}\mathbf{Hom}[\mathcal{C}]\langle$
| there exists a $\mathcal{C}$-arrow $\mathbf{S}f{\in}\mathbf{Hom}[\mathcal{C}]$ (aka the **source-arrow** of f) so that $\langle\ \mathbf{SS}f{=}\mathbf{S}f$ AND $\mathbf{TS}f{=}\mathbf{S}f\ \rangle$ AND
| there exists a $\mathcal{C}$-arrow $\mathbf{T}f{\in}\mathbf{Hom}[\mathcal{C}]$ (aka the **target-arrow** of f) so that $\langle\ \mathbf{ST}f{=}\mathbf{T}f$ AND $\mathbf{TT}f{=}\mathbf{T}f\ \rangle$
$\rangle$.
2) *Existence of identity-arrows*:
for every $\mathcal{C}$-arrow $f{\in}\mathbf{Hom}[\mathcal{C}]$

| there exists a $\mathcal{C}$-arrow $1_{\mathbf{S}f}\in\mathbf{Hom}[\mathcal{C}]$ (aka the **identity-arrow** of $\mathbf{S}f$) so that $\langle$ $\mathbf{S}1_{\mathbf{S}f}=\mathbf{S}f$ AND $\mathbf{T}1_{\mathbf{S}f}=\mathbf{S}f$ $\rangle$ AND
| there exists a $\mathcal{C}$-arrow $1_{\mathbf{T}f}\in\mathbf{Hom}[\mathcal{C}]$ (aka the **identity-arrow** of $\mathbf{T}f$) so that $\langle$ $\mathbf{S}1_{\mathbf{T}f}=\mathbf{T}f$ AND $\mathbf{T}1_{\mathbf{T}f}=\mathbf{T}f$ $\rangle$
$\rangle$.

3) *Existence of composite-arrows*:
for every $\mathcal{C}$-arrow $f\in\mathbf{Hom}[\mathcal{C}]$ and
for every $\mathcal{C}$-arrow $g\in\mathbf{Hom}[\mathcal{C}]\langle$
| IF $\mathbf{T}f=\mathbf{S}g$,
| THEN there exists a $\mathcal{C}$-arrow $gf\in\mathbf{Hom}[\mathcal{C}]$ (aka the **composite-arrow** of f **with** g) so that $\langle$
| | $\mathbf{S}gf=\mathbf{S}f$ AND
| | $\mathbf{T}gf=\mathbf{T}g$
| $\rangle$
$\rangle$.

PROPOSITION. **Identity**-arrows and **source**-arrows are the same. **Identity**-arrows and **target**-arrows are the same.
Let $\mathcal{C}$ be a category.
Let $f\in\mathbf{Hom}[\mathcal{C}]$ be a $\mathcal{C}$-arrow.

0) $1_{\mathbf{S}f}=\mathbf{S}f$.
1) $1_{\mathbf{T}f}=\mathbf{T}f$.
0′) The identity-arrow of the source-arrow of f is the source-arrow of f.
1′) The identity-arrow of the target-arrow of f is the target-arrow of f.

Sheaves keep track of **local-to-global** relationships between data in a way that ensures local-to-global consistency.

The idea is that we have a bunch of open sets of X stuffed into a topology $\tau_X \subseteq \mathcal{P}X$.
And we take an open set $U \subseteq X$.
And we take an open cover of U, say, the open cover $\{U_0, U_1\} \subseteq \tau_X$ made of *two* cover elements. Since $\{U_0, U_1\}$ covers U, then $U_0 \cup U_1 = U$.
On each cover element $U_i \in \{U_0, U_1\}$ there is a continuous map $f_i : U_i \longrightarrow \mathbf{R}$.
Since there are two cover elements (U_0 and U_1), and on each cover element there's a continuous map, then we have *two* continuous maps:
0) a continuous map $f_0 : U_0 \longrightarrow \mathbf{R}$ on U_0, and
1) a continuous map $f_1 : U_1 \longrightarrow \mathbf{R}$ on U_1.
And we want to look at all possible intersections of all cover elements.
So, we take all four interections of U_0 and U_1:
0) $U_0 \cap U_0$, which is just U_0,
1) $U_0 \cap U_1$,
2) $U_1 \cap U_0$, which is the same as $U_0 \cap U_1$,
3) $U_1 \cap U_1$, which is just U_1.
This yields *two extra* continuous maps:
0) the restriction of $f_0 : U_0 \longrightarrow \mathbf{R}$ to $U_0 \cap U_1$, which is denoted $f_0|_{U_0 \cap U_1} : U_0 \cap U_1 \longrightarrow \mathbf{R}$, and
1) the restriction of $f_1 : U_1 \longrightarrow \mathbf{R}$ to $U_0 \cap U_1$, which is denoted $f_1|_{U_0 \cap U_1} : U_0 \cap U_1 \longrightarrow \mathbf{R}$.
So, we started with two maps, f_0 and f_1, but now we have four:
0) $f_0 : U_0 \longrightarrow \mathbf{R}$,
1) $f_1 : U_1 \longrightarrow \mathbf{R}$,
2) $f_0|_{U_0 \cap U_1} : U_0 \cap U_1 \longrightarrow \mathbf{R}$, and
3) $f_1|_{U_0 \cap U_1} : U_0 \cap U_1 \longrightarrow \mathbf{R}$.
In general, the map $f_0 : U_0 \longrightarrow \mathbf{R}$ is different from the map $f_1 : U_1 \longrightarrow \mathbf{R}$.
And, in general, the restriction map $f_0|_{U_0 \cap U_1} : U_0 \cap U_1 \longrightarrow \mathbf{R}$ is different from the restriction map $f_1|_{U_0 \cap U_1} : U_0 \cap U_1 \longrightarrow \mathbf{R}$.

Now comes the good stuff.
We want to "glue" f_0 and f_1, which are defined on $U_0 \subseteq U$ and $U_1 \subseteq U$, into a **single** global map f defined on all of $U_0 \cup U_1$ (which is U).
But there isn't a *single* global map defined on all of $U_0 \cup U_1$: there are **two** global maps! Call them $f : U_0 \cup U_1 \longrightarrow \mathbf{R}$ and $g : U_0 \cup U_1 \longrightarrow \mathbf{R}$.
The global map $f : U_0 \cup U_1 \longrightarrow \mathbf{R}$ is defined piecewise, as follows.
0) For every x, IF x is in $U_0 - U_1$, THEN f maps x to $f_0[x]$.
1) For every x, IF x is in $U_1 - U_0$, THEN f maps x to $f_1[x]$.
2) For every x, IF x is in $U_0 \cap U_1$, THEN f maps x to $f_0|_{U_0 \cap U_1}[x]$.
The global map $g : U_0 \cup U_1 \longrightarrow \mathbf{R}$ is defined piecewise, as follows.
0) For every x, IF x is in $U_0 - U_1$, THEN g maps x to $f_0[x]$.
1) For every x, IF x is in $U_1 - U_0$, THEN g maps x to $f_1[x]$.
2) For every x, IF x is in $U_0 \cap U_1$, THEN g maps x to $f_1|_{U_0 \cap U_1}[x]$.
By definition, the global maps f and g agree on $U_0 - U_1$ and on $U_1 - U_0$, but they disagree on the intersection $U_0 \cap U_1$, because $f_0|_{U_0 \cap U_1}[x]$ need not equal $f_1|_{U_0 \cap U_1}[x]$ for $x \in U_0 \cap U_1$. (Recall that, in general, the restriction map $f_0|_{U_0 \cap U_1}$ is different from the restriction map $f_1|_{U_0 \cap U_1}$.)
But we can *demand* that f and g agree $U_0 \cap U_1$ too, and, in that case, f and g become the same map, ie. $f = g$.
So, if we want $f = g$, then we keep the piecewise definitions of f and g, and we add an extra condition:
For every x, IF x is in $U_0 \cap U_1$, THEN $f_0|_{U_0 \cap U_1}[x] = f_1|_{U_0 \cap U_1}[x]$.
This condition ensures that $f : U_0 \cup U_1 \longrightarrow \mathbf{R}$ and $g : U_0 \cup U_1 \longrightarrow \mathbf{R}$ are the same map, ie. $f = g$. This forces the **uniqueness** of a global map f.

And now we have a single **patchwork map** $f : U_0 \cup U_1 \longrightarrow \mathbf{R}$ defined on all of $U_0 \cup U_1$, constructed by "gluing" $f_0 : U_0 \longrightarrow \mathbf{R}$ and $f_1 : U_1 \longrightarrow \mathbf{R}$ and ensuring compatibility on the intersection $U_0 \cap U_1$.
Since $f_0 : U_0 \longrightarrow \mathbf{R}$ and $f_1 : U_1 \longrightarrow \mathbf{R}$ are continuous, then the patchwork map $f : U_0 \cup U_1 \longrightarrow \mathbf{R}$ is also continuous, but this requires proof.

DEFINITION. **Presheaves** (of abelian groups) on topological spaces.
Let (X, τ_X) be a topological space.
Let **Ab** be the category of abelian groups.
A **presheaf** $\mathcal{F}$ (of abelian groups) on the topological space (X, τ_X) IS
a **contravariant functor** $\mathcal{F}$ from τ_X to **Ab**, or equivalently
a **covariant functor** $\mathcal{F}$ from τ_X^{op} to **Ab**.
In detail.
0) For every τ_X arrow f
there exists an **Ab** arrow $\mathcal{F}f$ so that $\mathbf{S}\mathcal{F}f = \mathcal{F}\mathbf{S}f$ AND
there exists an **Ab** arrow $\mathcal{F}f$ so that $\mathbf{T}\mathcal{F}f = \mathcal{F}\mathbf{T}f$ AND
there exists an **Ab** arrow $\mathcal{F}1_{\mathbf{S}f}$ so that $\mathcal{F}1_{\mathbf{S}f} = 1_{\mathcal{F}\mathbf{S}f}$ AND
there exists an **Ab** arrow $\mathcal{F}1_{\mathbf{T}f}$ so that $\mathcal{F}1_{\mathbf{T}f} = 1_{\mathcal{F}\mathbf{T}f}$.
0) *Existence of arrows*:
for every τ_X arrow $f : U \longrightarrow V$
there exists an **Ab** arrow $\mathcal{F}f : \mathcal{F}U \longleftarrow \mathcal{F}V$.
1) *Composition compatibility*:
for every τ_X arrow $f : U \longrightarrow V$ and

for every τ_X arrow $g : V \longrightarrow W$
there exists an **Ab** arrow $\mathcal{F}gf : \mathcal{F}U \longleftarrow \mathcal{F}W$
so that $\mathcal{F}gf = \mathcal{F}f\mathcal{F}g$.

2) *Object/identity compatibility*:
for every τ_X identity arrow $1_U : U \longrightarrow U$
there exists an **Ab** identity arrow $\mathcal{F}1_U : \mathcal{F}U \longleftarrow \mathcal{F}U$
so that $\mathcal{F}1_U = 1_{\mathcal{F}U}$.

EXAMPLE.
LET τ_X be a category.
LET **Ab** be a category.
LET $f : U \longrightarrow V$ be a τ_X arrow.
LET $g : V \longrightarrow W$ be a τ_X arrow.
LET $\mathcal{F}$ be an **Ab**-presheaf on τ_X.
|
| SINCE τ_X is a category,
| AND $f : U \longrightarrow V$ is a τ_X arrow from U to V,
| AND $g : V \longrightarrow W$ is a τ_X arrow from V to W,
| AND $\mathbf{Tar}[f] = \mathbf{Src}[g]$,
| THEN, by the category axioms, there exists a τ_X arrow $gf : U \longrightarrow W$ from U to W.
|
| SINCE $f : U \longrightarrow V$ is a τ_X arrow from U to V,
| AND $g : V \longrightarrow W$ is a τ_X arrow from V to W,
| AND $gf : U \longrightarrow W$ is a τ_X arrow from U to W,
| AND $\mathcal{F}$ is an **Ab**-presheaf on τ_X,
| THEN, by presheaf arrow compatibility, there exists an **Ab** arrow $\mathcal{F}f : \mathcal{F}U \longleftarrow \mathcal{F}V$ to $\mathcal{F}U$ from $\mathcal{F}V$,
| AND, by presheaf arrow compatibility, there exists an **Ab** arrow $\mathcal{F}g : \mathcal{F}V \longleftarrow \mathcal{F}W$ to $\mathcal{F}V$ from $\mathcal{F}W$,
| AND, by presheaf arrow compatibility, there exists an **Ab** arrow $\mathcal{F}gf : \mathcal{F}U \longleftarrow \mathcal{F}W$ to $\mathcal{F}U$ from $\mathcal{F}W$.
|
| SINCE **Ab** is a category,
| AND $\mathcal{F}f : \mathcal{F}U \longleftarrow \mathcal{F}V$ is an **Ab** arrow to $\mathcal{F}U$ from $\mathcal{F}V$,
| AND $\mathcal{F}g : \mathcal{F}V \longleftarrow \mathcal{F}W$ is an **Ab** arrow to $\mathcal{F}V$ from $\mathcal{F}W$,
| AND $\mathbf{Tar}[\mathcal{F}g] = \mathbf{Src}[\mathcal{F}f]$,
| THEN, by the category axioms, there exists an **Ab** arrow $\mathcal{F}f\mathcal{F}g : \mathcal{F}U \longleftarrow \mathcal{F}W$ to $\mathcal{F}U$ from $\mathcal{F}W$.
|
| By presheaf composition compatibility, $\mathcal{F}gf = \mathcal{F}f\mathcal{F}g$.